PARADOXES IN SET THEORY

THAT TURN OUT NOT TO BE PARADOXES

James Holland

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Chapter I. Background

This document will use a lot of results about set theory as formulated in first order logic. As such, central results of first order logic are used extensively and are arguably the source of all of the “paradoxes” presented here. There’s a lot to say about first order logic and these central results. For any introduction to the subject, it is necessary to give some philosophy to explain both the perspective the field takes, and the assumptions of the field. After this, some of these central results can be presented and used in constructing various “paradoxes”. Only a basic introduction is given here, and so this chapter may be skipped by people who happen to know the material. Regardless, the hope is that this section will serve as a satisfactory introduction to the background material.

Section 1. *Philosophy*

An important distinction in logic is between the reasoning we use in the real world and the reasoning a certain subject allows. For example, we in the real world have the ability to conclude $a = c$ from $a = b$ and $b = c$. However if we consider only the sentential connectives there—‘and’ and ‘implies’—we cannot make the same conclusions. The logic no longer considers the meaning of equality, only the meaning of these sentential connectives.

To talk about this distinction, the reasoning we use in the real world will be called the *meta-theory* while the reasoning we study will be the *logic system*. The reasoning of a logic system is entirely formal, following from strings of symbols. Because both must be presented in writing here, the two are distinguished through colors. The strings of the logic system are written in red while meta-theoretic objects are in standard black. For example, in the meta-theory we might say “A and B”. In the logic system, we would write $A \wedge B$. That said, things standing for strings are meta-theoretic, and so written in black: for φ a formula, $(\varphi \wedge 1 + 1 = 4)$ is a string in the logic system.

An immediate question that pops up is “what kinds of logic systems do we care about?”. Fundamentally, the answer to this question relies on the speaker, but more generally we want to characterize actually valid reasoning, which means the logic of the meta-theory. Now it’s perhaps impossible to fully characterize the meta-theory, but it is possible to characterize fragments of it, like the sentential logic introduced before, and first order logic. Going further than this, however, requires addressing certain questions about the meta-theory and its connections with theories in these logic systems.

In particular, the question of our base level axioms come into question. A priori, there’s no guarantee that the world behaves in accordance with the axioms of ZFC or peano-arithmetic. In fact, we would need to reject them as part of the meta-theory if it turned out that either of these systems were inconsistent. Furthermore, constructions allowed by ZFC like $\mathbb{R}$ and $\mathbb{N}$ can be called into question if we reject certain axioms like the existence of $\mathbb{N}$. How then do we regard such statements as $|\mathbb{R}| > |\mathbb{N}|$? Is this a meta-theoretic fact, or is this better regarded as $|\mathbb{R}| > |\mathbb{N}|$, a statement in the logic system following from certain axioms? There are a few ways to address these concerns. Two major positions are presented here.

One stance is a purely formalist one. This view will neglect to say anything substantial about the meta-theory, taking only the most basic algorithmic reasoning needed for the study of logic for granted. The formalist approach then doesn’t connect the reality of the meta-theory with results of axioms like ZFC[i] in the logic system, and it in some sense ignores whether the theories we study are important at all. No commitments are made for whether the natural numbers $\mathbb{N}$ exist or whether a statement like “$|\mathbb{R}| > |\mathbb{N}|$” has any meaning in the meta-theory. But the formalist will

[i] To distinguish between the semantic content and the formulas, ZFC is the collection of meta-theoretic statements while ZFC is the collection of formulas that are supposed to represent the statements of ZFC. This distinction isn't maintained for other theories, since really only ZFC is assumed of the meta-theory beyond basic finitistic reasoning, and this only occurs for some results.

deny that $|\mathbb{R}| > |\mathbb{N}|$ has any semantic value. Instead, the formalist will view the statements about $\mathbb{N}$ or $\mathbb{R}$, for example, as merely symbols algorithmically changed from other symbols collectively called ZFC[ii]. So the results of theories in the logic system are seen purely as symbolic manipulation with no connection to the meta-theory. At best, a formalist will say the symbols in the logic system can be translated into arguments in the meta-theory where they should have been given in the first place. At worst, a formalist will say the symbols are devoid of content.

Another stance is a platonist or realist one. This view will hold that the results of axioms like ZFC and PA in the logic system do characterize a fragment of the meta-theory, and in fact the "real world". Not only is there a standard meaning of the statement $|\mathbb{R}| > |\mathbb{N}|$, but there is an actual fact of the matter, and we can learn such facts through study of theories in such logic systems. By and large, a platonist stance is held by mathematicians that want to claim that their conclusions are actually true and not merely derived from playing with symbols. Indeed most of mathematics is not done through symbolic algorithms like truth tables but instead through intuitions and clever constructions. That said, a platonist stance isn't strictly necessary, since often meta-theoretic arguments can be reformulated as symbolic ones and vice-versa. In this way the two stances are not incompatible.

This work will take more of a platonist stance. More precisely, ZFC is held as a collection of true statements about sets in the meta-theory, and this is used to reason about ZFC as presented symbolically. ZFC is assumed to be consistent generally, but results which rely on such assumptions will be marked. Sometimes these assumptions are abandoned by stating results in a more formal way to be more exact, especially when resolving these "paradoxes".

Section 2. *First Order Logic*

First order logic is the logic of quantifying over objects. There are two parts to this as with almost any logic system. Firstly there is a syntactic component ruling what can be said. Secondly there is a semantic component that gives meaning to these formulas. This separation is similar to the separation between the spelling of words in English, and the meaning of them. In this sense a proper setup of first order logic takes great deal of space explaining how to spell, and then how to interpret, but this will not be done here. Instead a brief overview will be given, and it will be fairly informal. There are a number of steps in this introduction. Continuing the natural language analogy, we need to

1. determine the alphabet we're using;
2. determine how to spell words with this alphabet;
3. determine how to "reason" with these words;
4. determine the meaning of these words;
5. connect spelling with meaning; and
6. reformulate this in a formalist way.

First order logic is not the only logic system one can use to study mathematics, but most other logic systems can be reformulated in terms of set theory with arguments that take place in first order logic. In fact, second order logic is sometimes called "set theory in sheep's clothing". Regardless, this section is about the framework in which the results of set theory are given—and results *about* set theory are about it in this framework. The paradoxes to be presented can more or less be resolved through careful analysis of how they are formulated in first order logic, and it is the aim of this section to give the framework for this logic system.

§ 2 A. The alphabet and its formulas

To start, the alphabet of first order logic is better regarded as a collection of alphabets that are all variations on a simpler alphabet. In particular, they all share the so called *logical symbols* given below that allow us to make basic formulas that are statements of equality and inequality: $x = y$, $v_3 = v_{10}$. From these basic statements—so called *atomic formulas*—we can build up larger formulas using simple rules. For things already determined to be formulas, we can connect them using formula connectives, or quantify them over some variable.

[ii] or whatever other foundation they are studied in.

Symbol	$\wedge$	$\vee$	$\neg$	$\rightarrow$	$\leftrightarrow$	$\forall$	$\exists$	$x, y, z, \ldots$	$=$	$($, $)$, $,$
Use	formula connectives					quantifiers		variables	equality	grammatical markers

So for φ and ψ already formulas, $(\varphi \wedge \psi)$, $\neg\varphi$, $\exists x(\varphi)$, and so forth are all formulas too. This allows us to build formulas like $\exists x \exists y(\neg x = y)$ and $(x = x \wedge \neg x = x)$. We cannot, however, make ordinary mathematical statements like $x = y + z$ or $\exists z(z \cdot x^{-1} \leq x + x + x + f(z, y))$ yet. To make such statements we need a bigger alphabet. In particular, we have the concept of a *signature* to expand the logical symbols with *non-logical symbols*.

2 A • 1. Definition

σ is called a signature if and only if σ is a collection of symbols that are divided into relation symbols and function symbols with the corresponding number of arguments.

Included in the function symbols are constant symbols, which are just functions with 0 arguments perhaps better thought of as constant functions. For example, rings and fields generally use a signature of just function symbols: $\{+, \cdot, 0, 1\}$. This expands the signature usually used with groups: $\{\cdot, 1\}$. Partial orders and graphs will use only relation symbols for the order and the edges. Most importantly for us, set theory uses the signature with only one element $\{\acute{\in}\}$[iii].

The rules for forming formulas change very little from when there were just logical symbols. Essentially, one just needs to respect the number of arguments for the relation and function symbols. So if f is a function symbol with two arguments, you can't write $f(x, y, z)$ or $f(t)$. The same applies to relation symbols. Building terms t_1, ..., t_n by composing function symbols and variables, we can let relations holding between terms—strings of the form $R(t_1, \cdots, t_n)$ or $t_1 = t_2$—be the basic building blocks of formulas. Then we can build the rest of the formulas in the same way as before.

Now we remark that often formulas are written in short-hand, meaning we don't include so many parantheses, and introduce symbols which are defined in the original signature. For example, $x \mathrel{\acute{\subseteq}} y$ can be defined by $\forall z(z \mathrel{\acute{\in}} x \rightarrow z \mathrel{\acute{\in}} y)$. Such defined notions affect nothing since they can be replaced by their defining formulas. In general, we're satisfied giving instructions for how to construct a formula as opposed to giving it explicitly. The same principle also holds for proofs.

§ 2 B. The proofs of formulas

With the notion of formula comes the notion of proof: a means of manipulating formulas. As mathematicians, the concept of proof should be fairly familiar. Giving a thorough, formal treatment of the subject is somewhat tedious, and so only an overview is given. Note that in setting up the proof system, we should be trying to emulate valid reasoning in the meta-theory, though there is no association of meaning with formulas yet. A priori, there's no reason we couldn't allow ourselves to conclude $\varphi \wedge \psi$ from $\varphi \vee \psi$—"both" from "at least one". So there is some careful setup required in what precisely is allowed—so called *logical axioms*—although from the perspective of a formalist it's unimportant. The following is an informal definition, omitting what precisely a logical axiom is.

2 B • 1. Definition

Let T be a collection of formulas, and φ a formula. T proves φ, or $T \vdash \varphi$, iff there is a sequence of formulas where every member

1. is a given assumption, i.e. a member of T; or
2. is a logical axiom, e.g. $x = x$ or $(\neg\neg\psi) \leftrightarrow \psi$; or
3. follows from previous ones by given rules of inference, e.g. ψ follows from φ and $\varphi \rightarrow \psi$.

For example, one can prove $\forall x \forall y(x + y = y + x)$ from the axioms of peano arithmetic, PA, which are then regarded as given assumptions in the above. A collection of formulas is generally called a *theory*. Note that the statement $T \vdash \varphi$

[iii] '$\acute{\in}$' is used because '$\in$' is reserved for the true, meta-theoretic membership relation. Arguably set theory uses many more symbols, e.g. '$\acute{\subseteq}$', '$\acute{\emptyset}$', and so forth. But these can be better regarded as short-hand for statements which use only '$\acute{\in}$' and '$\acute{=}$'.

for "there is a proof of φ from the formulas T" is a meta-theoretic one about the logic system. That said, one interesting result about set theory is that it can talk about such notions through formulas inside the logic system.

And as with formulas, it's rare to give proofs as just a sequence of formulas, because they are hard to read and comprehend. Even when annotated, it's hard to see at a glance that the formulas obey the definition. So often proofs are given as instructions for creating a proof rather than that sequence of formulas entirely. This perspective is useful when arguing in the meta-theory about proofs in the logic system.

This concludes the syntactic portion of first order logic, and now we will look towards interpreting these formulas, since thus far formulas are regarded merely as a bunch of markings formed in a certain way. The syntactic perspective will be picked back up in the last subsection, Subsection I.2.E, which will look at semantics as a syntactic notion.

§ 2 C. The semantics of formulas

Now we will move on to the semantics of first order logic, looking at how to interpret these formulas and reason from them in the meta-theory. In some sense the goal is to answer "what makes a formula true?". Answering this requires first fixing a context we ask the question in, and then we build up a notion of truth in just the same way we've built up formulas. The explanations given here relate somewhat back to the real world insofar as they assume that ZFC holds of sets in the meta-theory.

First the notion of a structure is introduced. These are in some sense where we evaluate truth. For example, when we ask whether the group operation $\cdot$ is commutative, we answer relative to some particular group. The question can be asked of any group, but the answer depends on the group we evaluate in. In the same way, we can ask questions in a fixed signature, but the answer depends on the structure.

2 C • 1. Definition

(ZFC) Let σ be a signature. M is a σ-structure or model if and only if $\mathsf{M} = \langle M, \varsigma \rangle$ where M is a set, and

1. For every n-place relation symbol $R \in \sigma$, there is one $R^{\mathsf{M}} \in \varsigma$ with $R^{\mathsf{M}} \subseteq M^n$; and
2. For every n-place function symbol $f \in \sigma$, there is one $f^{\mathsf{M}} \in \varsigma$ with $f^{\mathsf{M}} : M^n \to M$.

Intuitively, ς tells us how the model interprets the symbols of the signature σ, and the members of ς are the *interpretations* of the members of σ. For example, the signature $\sigma = \{\preccurlyeq\}$ has models which are really just any set equipped with a binary relation. For example $\langle \mathbb{N}, < \rangle$ is a $\{\preccurlyeq\}$-model, and so is any graph $\langle G, E \rangle$ where E is the edge relation of the graph. Under this definition, for any signature σ, any σ-model is also an $\emptyset$-model where there are no non-logical symbols, and the only statements are about equality[iv]. In fact for any σ-model is also a δ-model for any $\delta \subseteq \sigma$.

By following the construction of any given formula, this association of a symbol in σ with the interpretation in ς presents how to tell whether any given formula is true or false in a given structure in the natural way we read formulas. Note that there will always be a fact of the matter in any given structure of whether a formula is true or false in it, even if it isn't possible to determine practically. Explicitly, we have the following definition.

2 C • 2. Definition

(ZFC) Let σ be a signature with $R, f \in \sigma$ a relation and a function symbol. Let φ and ψ be formulas; and let M a model with various $m_i \in M$. Write

$$\begin{array}{lll}
\mathsf{M} \vDash R(m_1, \cdots, m_n) & \text{if and only if} & \langle m_1, \cdots, m_n \rangle \in R^{\mathsf{M}}, \\
\mathsf{M} \vDash m_1 = m_2 & \text{if and only if} & m_1 = m_2, \\
\mathsf{M} \vDash \varphi \wedge \psi & \text{if and only if} & \mathsf{M} \vDash \varphi \text{ and } \mathsf{M} \vDash \psi, \\
\mathsf{M} \vDash \neg\varphi & \text{if and only if} & \mathsf{M} \nvDash \varphi, \\
\mathsf{M} \vDash \forall x \varphi(x) & \text{if and only if} & \mathsf{M} \vDash \varphi(m) \text{ for every } m \in M, \\
\mathsf{M} \vDash \exists x \varphi(x) & \text{if and only if} & \mathsf{M} \vDash \varphi(m) \text{ for some } m \in M.
\end{array}$$

[iv] We can still say meaningful things in this language, but mostly this is about the number of things: $\exists x \forall y (x = y)$ will require that there is only one element, for example. Some systems also drop the need for equality, in which case there are no formulas without relation symbols.

Implicit in this is the ability to interpret terms in the signature, and this is done exactly as one would expect. For example, the interpretation of $f(m_1, g(m_2))$ is just $f^{\mathsf{M}}(m_1, g^{\mathsf{M}}(m_2))$. Also note that we are mixing formal symbols and non-formal ones, leaving the parameters implicit when needed. It's important to realize that parameters can only be used when we've fixed a particular model. Parameters—like $m_1, m_2 \in M$ in the above—are not symbols in the language, and so cannot be referenced in general. In some sense, parameters are used here merely to build up a notion of truth. With this concept firmly in place, notation will be slightly abused in the following ways.

2 C • 3. Definition

(ZFC) Let φ and ψ be formulas, and T a theory all in the signature σ. Let M be a σ-structure. Write

$\mathsf{M} \vDash T$	if and only if	$\mathsf{M} \vDash \theta$ for every $\theta \in T$.
$\varphi \vDash \psi$	if and only if	every σ-model M with $\mathsf{M} \vDash \varphi$ also has $\mathsf{M} \vDash \psi$.
$T \vDash \psi$	if and only if	every σ-model M with $\mathsf{M} \vDash T$ also has $\mathsf{M} \vDash \psi$.

For example, $(\varphi \wedge \psi) \vDash \varphi$, since any model $\mathsf{M} \vDash (\varphi \wedge \psi)$ has $\mathsf{M} \vDash \varphi$ by Definition 2 C • 2.

These definitions comprise all the semantics of first order logic, and they all take place all in the meta-theory, meaning that $\varphi \vDash \psi$ if there is a meta-theoretic argument about models of φ. Alternatively, it might be the case that all models of φ also model ψ merely by chance with no intelligible reason behind it. So far this situation hasn't been ruled out. It is up to the next subsection to dispel this possibility.

§ 2 D. Connecting syntax and semantics

We now have the basic setup for working in mathematics. On the one hand, we can symbolically manipulate our way to various formulas, and on the other, we can argue in the meta-theory about whether certain structures satisfy a given formula. The central question, however, is whether there is any connection between the two, that is, whether '$T \vdash \varphi$' and '$T \vDash \varphi$' have any relationship.

Clearly, we should have set up our proof system to be sound, that is to say that if $T \vdash \varphi$ then $T \vDash \varphi$. This way we aren't making any "mistakes" in our symbolic manipulations. Proving that any given proof system is in fact sound can be done fairly easily through meta-theoretic arguments about structures. Mostly this amounts to checking that each logical axiom and rule of inference holds in every model.

Quite a striking result in the study of first order logic is the completeness theorem which says that the converse also holds with our notion of proof. As an aside, this requires the axiom of choice of ZFC. Thus far we didn't need to assume the choice part of ZFC in the meta-theoretic parts of Subsection 2 C, but using ZFC is simpler than figuring out what is minimally needed. In general the following result holds so long as the signature σ can be well-ordered. In particular, the result holds for all finite signatures. Given that we will only be working in finite signatures, we could drop this assumption, but it's useful in stating later results to work in full generality.

2 D • 1. Theorem (Completeness)

(ZFC) Let T be a theory, and φ a formula. Therefore $T \vdash \varphi$ if and only if $T \vDash \varphi$.

This identifies the "accidental truth" of being true by chance in all models with the "justified truth" of proof. This also allows us to make conclusions from valid arguments in the meta-theory about models, and conclude that there are syntactic proofs of these results. Most important for our purposes is the fact that if $T \nvdash \varphi$, then $T \nvDash \varphi$. In particular, if T is *consistent*—meaning $T \nvdash (\varphi \wedge \neg\varphi)$—then there is a model of T. This connection between finite sequences of formulas and the existence of structures is somewhat surprising considering that structures can be very large. Furthering this relation between the finite and the infinite is the compactness theorem.

Given that proofs are finite, the compactness theorem for proofs can yield important results when paired with Completeness (2 D • 1).

2 D • 2. Theorem (Compactness)

(ZFC) Let T be a theory. Therefore T has a model if and only if each finite $\Delta \subseteq T$ has a model.

Proof .:.

If T has a model, then clearly every finite subset does too. But if T doesn't, then for any formula φ, $T \vDash (\varphi \wedge \neg\varphi)$, because no model $\mathsf{M} \vDash (\varphi \wedge \neg\varphi)$. By Completeness (2 D • 1), $T \vdash (\varphi \wedge \neg\varphi)$. Since proofs are finite, there is some finite subset $\Delta \subseteq T$ which contains all the formulas of T used in proving $(\varphi \wedge \neg\varphi)$. This finite subset then also has $\Delta \vdash (\varphi \wedge \neg\varphi)$, and so by soundness, $\Delta \vDash (\varphi \wedge \neg\varphi)$. Hence this finite subset of T can't have a model. ⊣

These two theorems are very useful for their ability to generate models. As noted above, consistent theories have models which say that they're true. This is the kind of black magic that allows us to form models of PA which aren't just $\mathbb{N}$. Adding to this black magic is the Löwenheim-Skolem theorem, which is the final theorem we need in the background of first order logic, and it again allows us to conclude the existence of models. This theorem will be the primary tool in the setup of Section II.1.

2 D • 3. Theorem (Löwenheim–Skolem)

(ZFC) Let σ be a signature, T be a σ-theory, and $\kappa \geq |\sigma|$ an infinite cardinal. Suppose there is a model $\mathsf{A} \vDash T$ with A infinite. Therefore there is a model $\mathsf{B} \vDash T$ with $|B| = \kappa$.

All the restrictions here are necessary. T needs to have an infinite model since sentences like $\exists y \forall x (x = y)$ will have models with only one element. κ needs to be infinite since theories like PA have no finite models.

$\kappa \geq |\sigma|$ is necessary since we could have a symbol c_r for every $r \in \mathbb{R}$. In this signature, there is the theory which calls all these constant symbols distinct: $T = \{c_r \neq c_s : r, s \in \mathbb{R} \text{ and } r \neq s\}$. T has an infinite model $\mathbb{R}$ where c_r is interpreted as r. Yet T requires any model to have at least $|\mathbb{R}|$ distinct elements. So no model of T can be countable.

These three theorems—I.2.D • 1, I.2.D • 2, and I.2.D • 3—comprise the main tools for generating paradoxes in that they relate limitations of finite proofs with the existence of structures. In this way, the limitations of first order logic allow the universe to be very flexible with the kinds of structures it can have. These kinds of strange structures can have many paradoxical properties. That said, we should perhaps be a little careful with precisely what we mean with "structure" and so forth, which relate to ideas in the meta-theory.

§ 2 E. Formalist first order logic

A skeptic to these set theoretic concepts will not be convinced by any proofs of Subsection 2 D, nor the concepts of Subsection 2 C, especially since all the theorems there require the axiom of choice. Moreover, even someone who accepts these set theoretic concepts will want to be convinced that the formal ZFC proves these theorems just as the meta-theoretic ZFC does. It is then our goal here to reformulate the results of these subsections in a purely symbolic way.

The treatment of first order logic given before takes us as existing in the real world, that is a place with trees, pigs, dirt, and—most important for logic—formulas. But in ZFC, there is no inherent notion of a formula. Instead, this needs to be a defined notion. We can accomplish this through encoding formulas into another defined notion, $\mathbb{N}$. In particular, we can code the symbols of our formal language through the following association that takes place in the meta-theory:

Symbol	$\wedge$	$\vee$	$\neg$	$\rightarrow$	$\leftrightarrow$	$\forall$	$\exists$	$=$	$($	$)$	$,$	$\acute{\in}$	the variable v_n
Symbol Code	1	2	3	4	5	6	7	8	9	10	11	12	$13 + n$

Although we haven't included the concept of a signature beyond set theory, this can be done in principle. It is omitted here because it slightly complicates things since the signature doesn't need to be countable. Now with the codes we have, formulas are just finite sequences of numbers that satisfy certain grammatical rules. Note that finiteness is also a defined notion. Again, we will skip the tedious detail. The reader just needs to believe that we can define these rules within ZFC, since ZFC can define what it means to be a sequence, and what it means to be the xth member in that sequence. Arithmetic can do this too by taking the sequence $n_1, \ldots, n_m$ of natural numbers to the number $2^{n_1+1}3^{n_2+1}\cdots p_m^{n_m+1}$ where p_m is the mth prime. But either way, this really just results in translating the rules ordinarily defined in terms of the symbols to the same rules in terms of their associated symbol codes.

Moving on, the same principle allows us to regard proofs as sequences of formulas obeying certain rules which can again be formally written down. Pulling this together, we get real-world formulas $\mathrm{formula}(x)$, and $y \vdash x$; defining what it means for x to be a formula, and to be provable from y respectively, although we no longer know that x is necessarily either of those things. For example, we can always translate $\langle 13, 8, 17\rangle$ into the formula $v_0 = v_4$. In general, for a real world formula φ, φ denotes the encoded formula, a sequence of natural numbers. Setting things up correctly, we will always get $ZFC \vdash \mathrm{formula}(\varphi)$, and similar statements. In essence, ZFC can confirm basic syntactic things we already know to be true about formulas and proofs.

By building up the definition using recursion, we can also define $\mathrm{models}(x, y)$[v] defining what it means for a formula y to be true in a structure x. Moreover, there are a whole host of other defined notions we can get using this kind of coding: the ones below for example.

Short-hand	Defining formula
$\mathrm{theory}(x)$	$\forall y(y \mathrel{\acute{\in}} x \to \mathrm{formula}(y))$
$\mathrm{satisfiable}(x)$	$\neg\exists z(x \vDash z \wedge z = \langle 3, 13, 8, 13\rangle)$
$\mathrm{structure}(x)$	$\exists M \exists \varepsilon(x = \langle M, \varepsilon\rangle \wedge \varepsilon \mathrel{\acute{\subseteq}} M \times M)$
$x \vDash y$	$\mathrm{theory}(x) \wedge \mathrm{formula}(y) \wedge \forall z\big[\mathrm{structure}(z) \wedge \forall t(t \mathrel{\acute{\in}} x \to \mathrm{models}(z, t)) \to \mathrm{models}(z, y)\big]$

With this, we can restate the same theorems as in Subsection 2 D. Moreover, the same sorts of proofs omitted will work by merely translating them into these defined notions. In this sense, these results have real symbolic proofs about the defined notion of proof. Note that the first turnstile below is black, because it is the real notion of proof in the real world.

2 E • 1. Theorem (Formal Completeness)

$ZFC \vdash \forall T\ \forall\varphi\, [\mathrm{theory}(T) \wedge \mathrm{formula}(\varphi) \to (T \vdash \varphi \leftrightarrow T \vDash \varphi)]$.

2 E • 2. Theorem (Formal Compactness)

$ZFC \vdash \forall T\, \big[\mathrm{theory}(T) \to \big(\mathrm{satisfiable}(T) \leftrightarrow \forall\Delta\, \big[\Delta \mathrel{\acute{\subseteq}} T \wedge |\Delta| < \aleph_0 \to \mathrm{satisfiable}(\Delta)\big]\big)\big]$.

The statement of the compactness theorem is even longer. To fit it properly write $\mathrm{modcard}(X, \kappa, Y)$ for

$$\kappa = |\kappa| \wedge \mathrm{structure}(X) \wedge \mathrm{theory}(Y) \wedge \exists x \exists e(X = \langle x, e\rangle \wedge |x| = \kappa) \wedge \forall f[f \mathrel{\acute{\in}} Y \to \mathrm{models}(X, f)].$$

This shorthand then reduces the compactness theorem to the still very long statement below.

2 E • 3. Theorem (Formal Löwenheim–Skolem)

$ZFC \vdash \forall T \forall\kappa\, \big[\exists A \exists\lambda\big(\kappa \geq \aleph_0 \wedge \lambda \geq \aleph_0 \wedge \mathrm{modcard}(A, \lambda, T)\big) \to \exists B\big(\mathrm{modcard}(B, \kappa, T)\big)\big]$.

Ultimately, this section will be useful in conjunction with our realist perspective in that there are two notions of models of set theory going around. Firstly, the sets in the real world satisfy ZFC and so ZFC. As such, all these formal theorems apply to an encoded version of set theory, ZFC. For instance, we could write $ZFC \vdash ZFC \vdash \mathrm{compactness}$.

[v] As with the meta-theoretic notion of '$\vDash$', we will later abuse notion and revert to $\vDash$ instead of models.

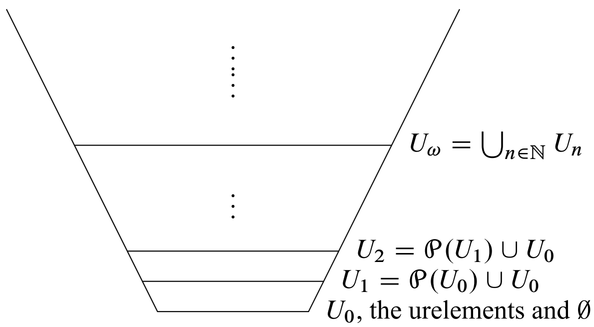
By formulating all of this inside arithmetic as noted at the beginning of this subsection, we could even do all this in arithmetic and get the absurd looking statement that PA $\vdash ZFC \vdash (ZFC \vdash \text{compactness})$ where we have to take the code of the code of ZFC and the compactness theorem in that second occurrence.

So in some sense, all of the work of Section 2 has been done with codes as statements about $\mathbb{N}$ within ZFC, and so it comprises just some of the results of the theory. The next section gives more results about this theory.

Section 3. *Set Theory*

Any decent introduction to set theory will need to give the axioms worked with. This will not be done here. Instead, this section will cover the major results and concepts used in the rest of the document which aren't assumed to be intuitive. For example, the definition of the ordinals and cardinals, and the structure of the universe of sets under ZFC.

To start out with, the general picture of the universe is given by two things. The first are so called *urelements*, things which aren't collections. These are things like dirt, clocks, formulas, and so on. The second are sets of these things and other sets. Explicitly, we can take the urelements along with Ø, and then keep taking powersets. We then get the following picture of the universe.



3 • 2. Figure: The universe

Not included in such a picture is the notion of a *class*. The issue brought forth by Russell's paradox is that not all properties yield a set of things the property applies to. For example, there is no set of all sets. Instead such collections are called "classes". The idea is that classes are a more general kind of collection than sets. So all sets are classes, but not all classes are sets. This idea is useful when talking about models of ZFC for two reasons. Firstly, when working in ZFC, all our models are sets from the real world perspective. As such, the collection of all sets inside a model is a set, though it's not a set *in the model*. Often from an outsider's perspective we can collect things together that the model doesn't think of as a set. From the model's perspective, we'll call such collections "classes". Examples of such classes are the class of all sets already mentioned, and the class of all ordinals, which have already been used in Figure 3 • 2 as subscripts of the U_αs.

Within ZFC, however, those notions of ordinals, cardinals, and so on are defined by formulas just in terms of $\acute{\in}$ and $=$. But reasoning in this way isn't very illuminating, and is difficult to write about properly. So to help with notation and explanation, the next two subsections will be written about an arbitrary model of ZFC.

§ 3 A. Defined notions: ordinals

In principle, all of this subsection could be written in a formal way. That said, it's difficult to explain the concepts in an intuitive way when all one can make reference to is formulas. So for this entire subsection, let $\mathbf{V} = \langle V, \in\rangle$ be an arbitrary model of ZFC. In this way, I can avoid writing "let $\wp$ be a function symbol and add to ZFC the axiom $Z = \wp(X) \leftrightarrow \forall Y(Y \acute{\in} Z \leftrightarrow \forall y(y \acute{\in} Y \to y \acute{\in} X))$, which defines a function by the other axioms of ZFC", and instead write "for any set X, define $\wp(X) := \{Y : Y \mathrel{\acute{\subseteq}} X\}$". Implicitly "set" ranges over the elements of V unless otherwise stated.

First we need to introduce the notion of ordinal numbers and sequences, which will always be indexed by ordinals. Ordinals are a kind of extension of natural numbers in that they are well-ordered. In fact, ordinals can be thought of as the canonical well-ordered sets. Explicitly, we have the following definition:

3 A • 1. Definition

($\langle V, \acute{\in}\rangle \vDash ZFC$) A set α is an ordinal, $\alpha \mathrel{\acute{\in}} \mathrm{Ord}$, iff $x \mathrel{\acute{\in}} \alpha \to x \mathrel{\acute{\subseteq}} \alpha$ and the members of α are well-ordered by $\acute{\in}$. Expanded, α is an ordinal iff

(1) $x \mathrel{\acute{\in}} \alpha \to x \mathrel{\acute{\subseteq}} \alpha$, i.e. α is *transitive*;

(2) $\acute{\in}$ is a linear order on α; and

(3) every subset of α has an $\acute{\in}$-minimal element.

This definition, of course, isn't very enlightening on a first reading. But there are a few immediate results that are very important for our purposes. For example, the definition allows us to conclude that any non-empty collection of ordinals has a $\acute{\in}$-least element[vi] just as any non-empty subset of the actual natural numbers has a least element. And this is what allows us to use induction on the ordinals in the same sort of way we do with the actual natural numbers, a process known as *transfinite induction*. Explicitly, the following theorem holds:

3 A • 2. Theorem (Transfinite Induction)

($\langle V, \acute{\in}\rangle \vDash ZFC$) Let P be a property. If for every $\alpha \mathrel{\acute{\in}} \mathrm{Ord}$, $\left[\forall \beta < \alpha\ P(\beta)\right] \to P(\alpha)$, then $P(\alpha)$ holds for every $\alpha \mathrel{\acute{\in}} \mathrm{Ord}$.

Proof .:.

Suppose not and let α be the least ordinal such that $\neg P(\alpha)$. Since α is the least such member, $\forall \beta < \alpha\ P(\beta)$. So by hypothesis $P(\alpha)$, a contradiction. $\dashv$

This well order property is a very useful one, but a better characterization is how these sets are constructed. In particular, we can start with $0 := \emptyset$, and then define the operations $x + 1 := x \cup \{x\}$, and $\sup \lambda := \bigcup_{\xi \acute{\in} \lambda} \xi$ when $\lambda \subseteq \mathrm{Ord}$ is a set of ordinals. Taking the supremum of such a λ still amounts to taking the least upper bound of all its members. Already this allows us to construct 0, $1 := \{0\}$, $2 := 1 + 1 = \{0, 1\}$, $3 := 2 + 1 = \{0, 1, 2\}$, and so on for all the natural numbers. Going further, we have the following characterization of ordinals.

3 A • 3. Theorem

($\langle V, \acute{\in}\rangle \vDash ZFC$) A set α is an ordinal iff

(1) $\alpha = 0$;

(2) $\alpha = \beta + 1$ for some $\beta \mathrel{\acute{\in}} \mathrm{Ord}$, i.e. α is a successor ordinal; or

(3) $\alpha = \sup \alpha$ with $\alpha \mathrel{\acute{\subseteq}} \mathrm{Ord}$, i.e. α is a limit ordinal.

This classifies all ordinals as either 0, a successor, or a limit, and by the axiom of infinity in ZFC, it allows us to construct sets like the least limit ordinal, ω, also known as the set of natural numbers.

With the ordinals introduced, the notion of a sequence is very important to all sorts of constructions.

3 A • 4. Definition

($\langle V, \acute{\in}\rangle \vDash ZFC$) A set x is a sequence iff there is an ordinal $\alpha \mathrel{\acute{\in}} \mathrm{Ord}$ with $x : \alpha \to V$. For any $\beta < \alpha$, the βth term, x_β, in the sequence x is really just $x(\beta)$. Denote the sequence x by $x = \langle x_\beta : \beta < \alpha \rangle$.

Combining Transfinite Induction (3 A • 2), Theorem 3 A • 3, and Definition 3 A • 4 allows us to recursively construct sets

[vi] The order on the ordinals, $\acute{\in}$, is also referred to with '$<$' to emphasize that it's an order.

based on what happens with each of the three kinds of ordinals, a process known as *Transfinite Recursion*. This tool is very powerful. For example, it allows infinite constructions like the U_α shown in Figure 3 • 2 under the meta-theoretic ZFC. And we can do the same process for ZFC as well.

3 A • 5. Definition

($\langle V, \acute{\in}\rangle \vDash ZFC$) Define by transfinite recursion the class of well-founded sets, $WF = \bigcup_{\alpha \acute{\in} \mathrm{Ord}} V_\alpha$:

$$\begin{aligned} V_0 &:= \emptyset \\ V_{\xi+1} &:= \wp(V_\xi) \\ V_\lambda &:= \bigcup_{\xi<\lambda} V_\xi, \quad \text{where } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Under ZFC, the class of well-founded sets actually comprises everything, as stated in the following theorem.

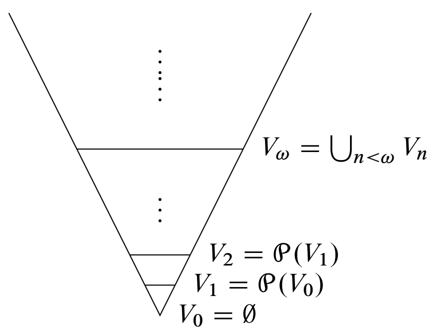
3 A • 6. Theorem

$ZFC \vdash \forall x(x \mathrel{\acute{\in}} WF)$. More precisely, $ZFC \vdash \forall x \exists \alpha(\alpha \mathrel{\acute{\in}} \mathrm{Ord} \wedge x \mathrel{\acute{\in}} V_\alpha)$.

Contained in this statement is that there are no urelements under ZFC, and everything is a set. This is a result of the axiom of extensionality:

$$\forall x \forall y(x = y \leftrightarrow \forall z(z \mathrel{\acute{\in}} x \leftrightarrow z \mathrel{\acute{\in}} y)).$$

Any urelement has no elements and so is equal to the empty set $\emptyset$. The resulting picture of the universe is then changed under ZFC. Urelements are mostly unnecessary for ZFC's purposes anyway. For the most part, any context in which urelements come up is one where we can just consider a set with the same number of elements, and then create the same structure on it.



3 A • 8. Figure: The universe of sets

This is just one result using ordinals and sequences. These concepts are very fundamental to set theory and mathematics in general. Ordinals are especially important, because they also contain all the cardinals.

§ 3 B. Defined notions: cardinality

The notion of "number of elements" is given by the notion of cardinality, renaming elements as ordinals.

3 B • 1. Definition

($\langle V, \acute{\in}\rangle \vDash ZFC$) Two sets X and Y are equipollent iff there is a bijection $f : X \to Y$.
The cardinality of X, $|X|$, is the least ordinal equipollent to X.
A set κ is a cardinal iff $|\kappa| = \kappa$.

Because we're assuming the axiom of choice, all sets have a cardinality. And because there are uncountable sets like $\mathbb{R}$, there are uncountable ordinals. To refer to such ordinals, we actually index the ordinals according to how large they are again using transfinite recursion:

$$\omega_0 := \omega$$
$$\omega_{\xi+1} \text{ is the least ordinal } \kappa \text{ with } |\kappa| > \omega_\xi.$$
$$\omega_\lambda := \sup\{\omega_\xi : \xi < \lambda\}, \text{ for } \lambda \text{ a limit ordinal.}$$

In particular, ω_1 is the least uncountable ordinal. Note that we can use these ordinals in the indexes: ω_ω and ω_{ω_1+1} for example. Further still, we can consider $\omega_{\omega_{\omega_{\cdots}}}$ defined as the supremum of the countable sequence where $\xi_0 = \omega$ and $\xi_{n+1} = \omega_{\xi_n}$. Such an ordinal has the odd property that $\xi = \omega_\xi$, demonstrating that the map $\alpha \mapsto \omega_\alpha$ has fixed points.

Now as a bit of notation, '$\aleph_\alpha$' will be written instead of 'ω_α' in the context of cardinality: $|\omega_\alpha| = \aleph_\alpha$. The two are equal as sets, but defined operations differ: $\omega \neq \omega + 1$ but $\aleph_0 = \aleph_0 + 1$ because $|\omega| = |\omega + 1|$. The sequence of $\aleph_\alpha$s exhausts all the infinite cardinals. In particular, the cardinals are $0, 1, 2, \cdots, \aleph_0, \aleph_1, \cdots, \aleph_\omega, \cdots$, which is just as long as the entire sequence of ordinals[vii].

§ 3 C. The natural numbers

We now return from considering only $\langle V, \acute{\in}\rangle \vDash$ ZFC, though we will still assume $\text{Con}(ZFC)$, that ZFC is consistent. Now recall Subsection 2 E, that we've encoded what it means to be a formula into the natural numbers. As such, statements about formulas, proofs, and so forth have been coded as statements about $\mathbb{N}$. As such, what ZFC can prove about formulas requires us to know what it thinks about $\mathbb{N}$. Now to better distinguish the two, the real world set of natural numbers is written '$\mathbb{N}$', while the set of natural numbers as a defined notion is written 'ω'. This subsection really only covers the distinction between $\mathbb{N}$ and ω^{M} for various models $\mathsf{M} \vDash ZFC$.

As stated in Subsection 3 A, ω^{M} is the least limit ordinal of M. However, this need not be the real natural numbers, nor even isomorphic to them, as the following theorem shows.

3 C • 1. Theorem

(ZFC, $\text{Con}(ZFC)$) There is a model $\mathsf{M} \vDash ZFC$ such that the real natural numbers $\mathbb{N} \neq \omega^{\mathsf{M}}$.

Proof .:.

Let c be a constant symbol, and consider the theory

$$T = ZFC \cup \{c \mathrel{\acute{\in}} \omega\} \cup \{n < c : n \in \mathbb{N}\},$$

where n is really just $1 + 1 + \cdots + 1$, adding together n 1s. Any finite subset $\Delta \subseteq T$ will have

$$\Delta \subseteq ZFC \cup \{c \mathrel{\acute{\in}} \omega\} \cup \{0 < c, 1 < c, \cdots, n < c\},$$

for some $n \in \mathbb{N}$. Note that $ZFC \vdash n < (n+1) \mathrel{\acute{\in}} \omega$, so that $\Delta \cup \{c = n+1\}$ has a model. To see this, by $\text{Con}(ZFC)$, ZFC has a model $\mathsf{W} \vDash ZFC$, the expansion $\mathsf{W}' \vDash \Delta$, interpreting c as $n + 1^{\mathsf{W}}$.

By Compactness (2 D • 2), T has a model M where then $\mathsf{M} \vDash c \mathrel{\acute{\in}} \omega \wedge n < c$ for every $n \in \mathbb{N}$. As a result, the order-type of ω^{M} must be different than that of $\mathbb{N}$. $\dashv$

This theorem motivates the following distinction in the kinds of models of ZFC.

3 C • 2. Definition

(ZFC) A model $\mathsf{M} \vDash ZFC$ is <u>ω-standard</u> iff $\omega^{\mathsf{M}} = \mathbb{N}$, the real world natural numbers.

It makes no difference if we say the two are isomorphic to as opposed to equal, since we can just consider the isomorphic model that contains the real natural numbers. Now although we can't prove the existence of ω-standard models, as we'll see in Section II.3, we can still make the distinction. Regardless, we can prove the existence of ω-non-standard models as Theorem 3 C • 1 demonstrates, given the consistency of ZFC. Moreover, the proof of Theorem 3 C • 1 shows that even if is misinterpreted, we can still find a copy of $\mathbb{N}$ inside, and in fact we have the following result.

[vii] As defined here, the ordinals wouldn't be a sequence since Ord $\mathrel{\acute{\notin}}$ Ord, but really we can just say something like "for every ordinal α, the sequence $\langle \aleph_\beta : \beta < \alpha\rangle$ exists and $\aleph_\xi \neq \aleph_\beta$ for $\xi \neq \beta < \alpha$".

3 C • 3. Theorem

(ZFC, Con(ZFC)) Let M be ω-non-standard. Therefore the real natural numbers $\mathbb{N}$ are an initial segment of ω^{M}. In fact, ω^{M} has order-type $\omega + (\omega^* + \omega) \cdot \xi$ where ξ is some dense linear order.

In essence, any improperly interpreted ω looks like $\mathbb{N}$ followed by a bunch of copies of $\mathbb{Z}$. In particular, if the misinterpretation of ω is countable, then it looks like $\mathbb{N}$ followed by $\mathbb{Q}$ copies of $\mathbb{Z}$.

Now so far, we've stated things informally about how the natural numbers can be interpreted in ZFC, but it's not entirely clear yet what is meant by this. The two theories of arithmetic and set theory are in different signatures, so we first need to have a translation between the two. This allows us to make the notion of 'interpretation' formal between any two theories.

3 C • 4. Definition

Let τ, π be two finite signatures. Let T be a τ-theory, and P be a π-theory. We say $\underline{P \text{ can be interpreted in } T}$ iff there is a function tolk from π-formulas to τ-formulas such that for any π-formulas φ and ψ,

1. if $P \vdash \varphi$, then $T \vdash \text{tolk}(\varphi)$;
2. $T \vdash (\text{tolk}(\neg\varphi) \leftrightarrow \neg\text{tolk}(\varphi))$ and $T \vdash \left[\text{tolk}(\varphi \wedge \psi) \leftrightarrow (\text{tolk}(\varphi) \wedge \text{tolk}(\psi))\right]$; and
3. tolk is primitive recursive (i.e. computable in a very basic way).

Basically, tolk gets the basic sentential symbols $\wedge$, $\vee$, $\rightarrow$, $\neg$, and so on correct—at least according to T. In essence, P can be interpreted in T if we can make P as just a part of the theory T after some basic translations. Through the use of ordinals and so forth, this allows us to interpret statements in PA and about arithmetic in general in ZFC through basic rules. So we get the following result.

3 C • 5. Theorem

PA can be interpreted in ZFC.

Statements like this are what allows so much of mathematics to be carried out through ZFC and set theory in general.

Of course there is more to say about set theory, but this section has already gone over more than what will be used in this document. And while this concludes the introduction of Chapter I , the discussion and theory of this material will continue in Chapter II and Appendix A[viii].

[viii] which should be expected, since that's just the rest of the document.

Chapter II. The Paradoxes

Thoralf Skolem, who is the same Skolem from Löwenheim–Skolem (2 D • 3), used the result now known as *Skolem's paradox* to conclude that set theory wouldn't—or perhaps *shouldn't*—serve as a "satisfactory ultimate foundation of mathematics" [5]. Given the unavoidable nature of Skolem's paradox and other similar paradoxical results, there does seem to be some content in this claim, however it is interpreted. That said, more nuanced understandings can help reclassify such results as straight forward instead of paradoxical.

There are a few paradoxes presented here of varying degrees of setup and background. Most of these paradoxes will consist of a setup which goes into the statement of the result and why it is paradoxical, and a resolution which attempts to address the counter-intuititive result and make it more understandable. Almost all of these paradoxes arise from ambiguity about where we're evaluating truth or where the statements are made. In particular, there are ambiguities about what a set is. This is not a complete overview of all paradoxes in set theory—if there even could be—but I thought there were a few here that deserved some recognition and explanation.

Section 4. *Skolem's Paradox*

Put naively, Skolem's paradox is that there are countable models of set theory which contain uncountable sets. Of course, there is a subtlety to avoid this contradiction. Like all these paradoxes, the issue arises from ambiguity about where we're evaluating truth or where the statements are made. In this case, we will look at cardinality from two perspectives.

§ 4 A. Setup to Skolem's paradox

Arguably the simplest to setup, Skolem's paradox only requires some working knowledge of Section 2 and some background understanding of cardinality. As stated before, we will be assuming the consistency of ZFC, $\mathrm{Con}(ZFC)$. In light of the theorems of Section 2, there is a countable model of ZFC in the real world universe of sets. This should already be fairly odd to think about: a countable model of all of set theory which is supposed to provide a basis for math? What about uncountable things? Indeed Skolem's paradox is more or less expressing this skepticism.

4 A • 1. Result (Skolem's paradox)

(ZFC, $\mathrm{Con}(ZFC)$) There is a structure $\mathbf{U} = \langle U, \varepsilon \rangle$ such that

(1) $\mathbf{U} \vDash ZFC$;

(2) $|U| = \aleph_0$; and

(3) there is an $R \in U$ where $\mathbf{U} \vDash |R| > \aleph_0$.

Proof .:.

To show (1) and (2), note that the consistency of ZFC yields by Completeness (2 D • 1) a model $\mathbf{M} \vDash ZFC$. Any model of ZFC is clearly infinite—in particular all the natural numbers are distinct elements. So by Löwenheim–Skolem (2 D • 3) there is a model $\mathbf{U} \vDash ZFC$ of cardinality $\aleph_0$.

With this model $\mathbf{U}$, there is some $R = \mathbb{R}^{\mathbf{U}} \in U$. Now since $ZFC \vDash |\mathbb{R}| > \aleph_0$ and $\mathbf{U} \vDash ZFC$, it follows that $\mathbf{U} \vDash |R| > \aleph_0$. ⊣

The paradox is that it seems our supposedly countable structure contains uncountably many things! So what sort of trickery is going on here?

A first reaction might be to dismiss it by saying that such models must have drastically misinterpreted the relation $\acute{\in}$ in axioms of ZFC just as ω-non-standard models have misinterpreted $\mathbb{N}$. But we can't rule out in general that such models interpret $\acute{\in}$ as the real world membership relation $\in$ and see all the real world members of R. Put more succinctly, it may be possible[i] that $R \subseteq U$, not merely a misinterpretation of a set.

§ 4 B. Resolution of Skolem's paradox

To make a long story short, the structures described in Skolem's paradox (4 A • 1) get it wrong, but the way they get it wrong can be somewhat interesting. To start with, even if the elements of a model $\mathsf{M} \vDash ZFC$ are not sets in the usual sense[ii] we can still consider the cardinality in the same sort of way we normally would.

4 B • 1. Definition

(ZFC) Let $\mathsf{M} = \langle M, \varepsilon \rangle$ be a $\{\acute{\in}\}$-structure, and $X \in M$. Define $\|X\| := \left|\left\{x \in M : \mathsf{M} \vDash x \mathrel{\acute{\in}} X\right\}\right|$.

In some sense, $\|X\|$ is the actual cardinality of X despite what M *thinks* the cardinality is. So this is the paradox: in Skolem's paradox (4 A • 1), $\|R\| = \aleph_0$ but $\mathsf{U} \vDash |R| > \aleph_0$. If we run through Cantor's diagonalization argument to show $|\mathbb{R}| > \aleph_0$, we should be able to explicitly and constructively disprove that $\|R\| = \aleph_0$. So how is this possible? The issue is that such constructions wouldn't exist in U.

Recall Definition 3 B • 1 the definition of cardinality. Evaluating cardinality requires the ability to rename elements through a bijection. The resolution then amounts to recognizing that U might not contain all the actual bijections despite satisfying the axioms ZFC. In an analogous situation, a non-abelian group $\mathsf{G} = \langle G, \cdot \rangle$ might have a non-trivial center $Z(\mathsf{G}) < \mathsf{G}$ which is then abelian. Despite the fact that both $Z(\mathsf{G}) \vDash$ Group Axioms and $\mathsf{G} \vDash$ Group Axioms, they disagree about whether $\cdot$ is commutative or not. Even though $Z(\mathsf{G})$ interprets $\cdot$ in the same way as G, it doesn't see the same objects. $Z(\mathsf{G})$ believes that everything commutes, because implicitly "everything" is just "everything I contain". G is able to see the problem elements and realize that the operation isn't commutative. In just the same way, the actual set theoretic universe V may contain models of ZFC that don't see these problem bijections.

In this way, U can go on believing that R is $\mathbb{R}$, because it closes its eyes, covers its ears, and shouts "la la la la I can't hear you la la la la" when you present the actual bijection between R and $\mathbb{N}$. Furthermore for any definable ordinal α, any set model $\mathsf{M} \vDash ZFC$ with cardinality $|M| = \aleph_\alpha$ will have this issue, because the model will then still believe there are sets of arbitrarily larger cardinality like $\aleph^{\mathsf{M}}_{\alpha+1}$. In this way, models of ZFC only look at things which make themselves look big compared to their members.

Section 5. *Illfounded Models of Foundation*

The existence of illfounded models of ZFC again presents a certain degree of freedom that first order logic can't restrict. Although it deals with concepts less familiar than Skolem's paradox, the resulting conflict seems a fair bit more important. Skolem's paradox presented a freedom that allows a conflict between cardinality from an outside perspective, and cardinality from an inside perspective. In the same sort of way, illfounded models allow a conflict about the well-foundedness of membership. As a result, perhaps the sort of visualization given in Figure 3 A • 8 isn't entirely accurate.

[i] Under certain assumptions we can rule this out, but those assumptions are not clearly true of real world sets.

[ii] This is trivially possible if we consider a universe of sets U and replace $\{\emptyset\} \in U$ with, say, an octopus. All the relationships would stay fixed, but the universe itself wouldn't contain only sets. So $\emptyset \mathrel{\acute{\in}}$ 🐙 and $\{$🐙$\} = \langle \emptyset, \emptyset \rangle$ would be true for example, although both are false in the real world: $\emptyset \notin$ 🐙.

§ 5 A. Existence of illfounded models

The main idea of this paradox is that a formula and property, that are equivalent in the meta theory, aren't actually equivalent logically. To make matters worse, it's not just that this formula is a bad example: there is actually no set of formulas that is logically equivalent to the concept. Not only does this demonstrate the weakness of formulas to characterize structures, but it also demonstrates the weakness of our attempts to found the meta theory in first order set theories.

Recall that a relation R is called *well-founded* iff there is no infinite R-descending sequence. This is a property that looks like it could be written easily in first order logic:

$$\neg\exists x_0 \exists x_1 \cdots \left(\bigwedge_{i \in \mathbb{N}} x_{i+1} \ R \ x_i \right).$$

The only issue is that this isn't a formula since it is infinitely long. Instead, we can use the concepts of set theory to say that any infinite sequence S, like $\langle x_n : n \in \mathbb{N} \rangle$, is not an infinite R-descending sequence:

$$\neg \forall x \mathrel{\acute{\in}} S \ \exists y \mathrel{\acute{\in}} S \ (y \ R \ x).$$

Indeed if there were such a collection, we'd have that R isn't well-founded. As a result, the two are equivalent for real world sets V. To be very formal, the axiom of foundation—attempting to say that $\acute{\in}$ is well-founded—is the following:

$$\forall S (S \neq \emptyset \rightarrow \neg \forall x \mathrel{\acute{\in}} S \ \exists y \mathrel{\acute{\in}} S \ (y \mathrel{\acute{\in}} x)).$$

But the existence of illfounded models is guaranteed by Compactness (2 D • 2). Its proof is similar to the existence of ω-non-standard models Theorem 3 C • 1. Again, we will be assuming the consistency of ZFC although the method applies to *any* formulation of set theory that contains the natural numbers.

5 A • 1. Result

(ZFC, Con(ZFC)) There is a model $\mathbf{I} = \langle I, \varepsilon \rangle \vDash ZFC$ such that ε is not well-founded.

Proof .:.

For each $n \in \mathbb{N}$, let c_n be a constant symbol. Now consider the theory in the language $\{\acute{\in}, c_0, c_1, \cdots\}$

$$T = ZFC \cup \{c_{n+1} \mathrel{\acute{\in}} c_n : n \in \mathbb{N}\}.$$

Any model $\mathbf{U} = \langle A, \eta, a_0, a_1, \cdots \rangle \vDash T$ will ensure η isn't well-founded because $\{a_n : n \in \mathbb{N}\}$ has no minimal element. Explicitly, for any a_n, we know $a_{n+1} \ \eta \ a_n$ because $\mathbf{U} \vDash c_{n+1} \mathrel{\acute{\in}} c_n$. Since $a_{n+1} \in \{a_i : i \in \mathbb{N}\}$, a_n isn't η-minimal in this collection.

Claim 1

T has a model.

Proof .:.

By Compactness (2 D • 2), it suffices to show every finite subset $\Delta \subseteq T$ has a model. Note that any finite subset $\Delta \subseteq T$ is such that

$$\Delta \subseteq ZFC \cup \{c_n \mathrel{\acute{\in}} c_{n-1}, \cdots, c_2 \mathrel{\acute{\in}} c_1, c_1 \mathrel{\acute{\in}} c_0\}.$$

Hence any model of ZFC can be expanded to one of Δ by interpretting c_n as 0, c_{n-1} as 1, ..., c_1 as $n-1$, and c_0 as n. In other words, let $\mathbf{W} \vDash ZFC$, and consider the expansion $\mathbf{W}'$ with the interpretations $c_i \mapsto (n-i)^{\mathbf{W}}$ if $i \leq n$, and anything else otherwise—$0^{\mathbf{W}}$ for example. It's clear $\mathbf{W}' \vDash \Delta$. As Δ was arbitrary, T also has a model by Compactness (2 D • 2). ⊣

So let $\mathbf{I} = \langle I, \varepsilon, i_0, i_1, \ldots \rangle \vDash T$. By the argument given above, ε is not well-founded, yet $\mathbf{I} \vDash ZFC$. ⊣

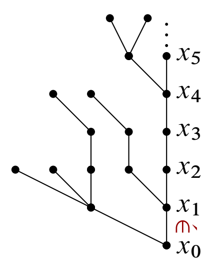
This appears to be a blatant contradiction: there is an infinite ε-descending sequence of the elements of I in Result 5 A • 1, $\mathbf{I}$ satisfies the axiom of foundation which is supposed to state that there can be no such collection. So even though the two formulations of well-foundedness are equivalent in the real world, they somehow can differ in first order logic. How can this happen?

§ 5 B. Resolution of the conflict

When we look closely at the axiom of foundation, it is a statement about sets. That is to say the collections we talk about are sets as interpreted by $\mathtt{I}$. So when say that the axiom of foundation rules out an infinite decreasing $\acute{\in}$-sequence, it really rules out *sets* of infinitely decreasing $\acute{\in}$-sequences. So another way to view the axiom is that only classes can contain infinitely decreasing $\acute{\in}$-sequences. Putting the axiom this way is a little more conducive to believing that this situation can happen.

Now there are a few ways to try to attack this response, since naively we should be able to just find a set F witnessing that a model $\mathtt{I} \vDash ZFC$ is illfounded. One way is to appeal to the axiom of choice and recursion in ZFC. Start out with the initial $x_0 \in F$ and for each $n > 0$, because $\mathtt{I}$ is illfounded, $\mathtt{I} \vDash \exists x(x \acute{\in} x_n)$ and so by choice we can let $x_{n+1} \in I$. By recursion and replacement, $\{x_n : n \in \mathbb{N}\} \in I$, right? The issue with this is two-fold.

Firstly, the guarantee that there is *some* element in each x_n doesn't tell us the resulting choice of x_{n+1} will lead to an infinite $\acute{\in}$-decreasing sequence. We can consider the reverse $\acute{\in}$-tree of $\mathtt{I}$ as in II.2.B • 1. Figure 3 below. Note that although there is an infinite branch, there are also many finite branches which would violate that $\mathtt{I} \vDash \exists x(x \acute{\in} x_n)$ if we improperly let x_n be on one of these branches. In fact, we will be unable to always choose properly because the branch as a whole isn't seen by $\mathtt{I}$. In the real world, we could apply König's theorem through the axiom of choice. But despite the fact that ZFC proves König's theorem, this alone does not guarantee that this infinite path is seen by $\mathtt{I}$, because König's theorem assumes the tree itself exists. But the membership relation isn't a set for any model of ZFC. But more succinctly, $\mathtt{I}$ can't see all of itself: $I \notin I$.



5 B • 2. Figure: An illfounded $\acute{\in}$-graph

The second problem with this is that we can't just define the proper induction in general to get the sequence of $c_n^{\mathtt{I}}$s. Although we have meta-theoretically defined a function $n \mapsto c_n$, $\mathtt{I}$ doesn't know about this association: with no context, the c_n are merely symbols. So this approach needs to answer "how do we *define* such a sequence?". And the answer is that we can't. There are infinitely many of these symbols, and so we can't just form a formula to define the set like if the sequence were finite: for each $n \in \mathbb{N}$, $\exists x(x = \langle c_0, c_1, \cdots, c_n\rangle)$ is—short-hand for—a formula, but $\exists x(x = \langle c_0, c_1, \cdots\rangle)$ is not.

The result is that the axiom of foundation cannot ensure that our models are actually well-founded. The axiom of foundation rules out all finite loops

$$ZFC \vdash \neg\exists x_0 \cdots \exists x_n (x_0 \acute{\in} x_1 \acute{\in} \cdots \acute{\in} x_n \acute{\in} x_0),$$

but cannot properly look at infinite ones, because—just like with Section 4—the models don't necessarily contain all the things the actual universe does.

In fact, no axiom excluding a contradiction can truly ensure that membership is well-founded. The proof of Result 5 A • 1 will apply as long as the models contain what they believe is $\mathbb{N}$, as guaranteed by the axiom of infinity. The strength of the compactness theorem is both a power of set theory and first order logic, demonstrating the existence of certain structures, and a weakness, demonstrating how little can be expressed. Even something so tangible as being well-founded can't be expressed properly in first order logic, as this section demonstrates[iii].

[iii] There is some redemption for first order logic, however, with the idea of realizing and omitting *types*. A well-founded model is just one which never has all of the formulas of $\{v_1 \acute{\in} v_0, v_2 \acute{\in} v_1, \cdots\}$ true, it *omits* that *type*. Similarly, the standard model of the natural numbers is just the unique model of PA which omits $\{x \neq 0, x \neq 1, x \neq 2, \cdots\}$. The reason this concept allows us to better distinguish structures is that we can

These problems that working in first order logic creates might motivate one to abandon this in favor of a more solid foundation in something like second order logic to avoid misinterpreting $\mathbb{N}$ or getting cardinality wrong, but the meta-theory used there requires making these set theoretic concepts clear and unambiguous. In particular, we need to already have a robust concept of the infinite, which it seems has only come from set theory as formulated in first order logic. And in light of Section 4 and Section 5 as well as independence results like that of the continuum hypothesis, the relativity of these issues means this first order formulation is inadequate to nail down these concepts properly. The second order formulation effectively ignores the issue, saying that it takes a stance yet is unable to confirm one for us, because of this issue from first order logic.

Section 6. Reflection, Compactness, and Gödel

The issues presented so far have been expressing a kind of freedom to find strange structures, and have mostly used just the results of Section 2. This section marks a turning point away from this method of generating paradoxes and a move towards issues in set theory proper. The concepts introduced in Section 5, particularly around interpreting the natural numbers, will be investigated further. These kinds of paradoxes will be the most technical, relying on coding of formulas and proofs in set theory as in Subsection 2 E. And unlike the previous two, this section will contain more than just a setup and resolution, presenting some background for three paradoxes relating to the principle of reflection and Gödel's second incompleteness theorem.

§ 6 A. Gödel's incompleteness theorems

Gödel's first incompleteness theorem is one of the more famous results in mathematics, although its precise statement isn't nearly as well known. In basic terms, there are things unprovable but true. Immediately this seems to conflict with Completeness (2 D • 1), but formulated more precisely it doesn't really: "truth" here refers to truth about the real world natural numbers, and not about all structures. Still, there is a conflict which says that there is a model which disagrees with the natural numbers, despite the fact that both satisfy PA. Gödel's second incompleteness theorem gives the even more surprising result that there will be models of PA saying that PA is inconsistent. Before getting too deep into these paradoxical results, however, we must state more precisely what Gödel's theorems are.

The first theorem says that some statements true of the real world natural numbers, $\langle \mathbb{N}, 0, 1, +, \cdot \rangle$, aren't provable from PA, and in fact every intelligible, consistent theory of arithmetic has this property. Gödel's second incompleteness theorem shows that the consistency of the theory is one of these statements. As in Subsection 2 E, the statement "such-and-such theory is consistent" is given by coding formulas into numbers, and then sequences of formulas also into numbers. Given that we can interpret $\mathbb{N}$ in ZFC, the result expands to ZFC itself, yielding that ZFC cannot prove its own consistency: $\mathrm{Con}(ZFC)$. More precisely, we have the following theorem, using the following non-standard definition for the sake of space:

6 A • 1. Definition

Let T be a theory in some finite signature τ. T is <u>ACC</u> (arithmetical, consistent, and computable) iff
(1) PA can be interpreted in T;
(2) T is consistent; and
(3) It is computable whether any formula φ is in T or not.

6 A • 2. Theorem (First Incompleteness Theorem)

Let T be ACC. Therefore there is a formula ψ such that $T \nvdash \psi$ and $T \nvdash \neg\psi$.

reference variables infinitely many times, unlike with separate sentences, which can only reference finitely many variables at a time.

Formulating the theorem in this way shows just how broad the result it. For example, PA clearly falls under this label. Hence there are formulas which are neither provable, nor disprovable from PA. But even if we add one of these formulas to PA, the resulting theory again still has independent formulas.

Indirectly, we could instead look at proofs from ZFC. For any formula φ in the language of arithmetic $\{0, 1, +, \cdot\}$, we can replace these symbols with their defining notions in ZFC, and restrict the quantifiers of φ to $\mathbb{N}$. We can then look at the resulting formula ψ and say that the original φ is deduced from ZFC iff we give a proof of ψ from ZFC. The result is that this new system—translating and proving in ZFC—still meets all the requirements if we assume $\mathrm{Con}(ZFC)$. Hence there are still statements about arithmetic that are neither provable, nor disprovable in this system.

Now we've codified consistency as statements about arithmetic, and it turns out that these arithmetical formulas are examples of such independent results:

6 A • 3. Theorem (Second Incompleteness Theorem)

Let T be ACC. Therefore $T \nvdash \mathrm{Con}(T)$ and $T \nvdash \neg\,\mathrm{Con}(T)$.

Note that this theorem depends on how we codify consistency. So if we use a different association in the meta-theory, we get different statements that are also independent. But implicit in the statement of the theorem is that the statement of consistency has been coded into the natural numbers. Given that we can do this with ZFC, we get the first paradoxical result of this section.

§ 6 B. Models of inconsistency

Given Completeness (2 D • 1) and assuming that ZFC is consistent, Gödel's Second Incompleteness Theorem (6 A • 3) would seem to imply that there are models of ZFC that say ZFC is consistent, and models that say ZFC is inconsistent. What do we make of such models? Surely they must realize that they satisfy ZFC themselves, right?

6 B • 1. Result

(ZFC, $\mathrm{Con}(ZFC)$) There is a model $\mathsf{W} \vDash ZFC$ such that $\mathsf{W} \vDash \neg\,\mathrm{Con}(ZFC)$.

Proof .:.

By Second Incompleteness Theorem (6 A • 3), $ZFC \nvdash \mathrm{Con}(ZFC)$. By Completeness (2 D • 1), there is a model $\mathsf{W} \vDash ZFC$ such that $\mathsf{W} \nvDash \mathrm{Con}(ZFC)$, whence $\mathsf{W} \vDash \neg\,\mathrm{Con}(ZFC)$. ⊣

Even worse, because of how we have coded the idea of consistency, such a W will say that it has a proof of a contradiction from ZFC! But how can this be possible given that we've assumed that ZFC is actually consistent? To resolve this conflict, we need to investigate the difference between ZFC and ZFC^{W}, and as we've coded this into arithmetic, the difference between $\mathbb{N}$ and $\mathbb{N}^{\mathsf{W}}$.

§ 6 C. Resolution to models of inconsistency

The difference between $\mathbb{N}$ and the interpretation of $\mathbb{N}$ has already been introduced with the notion of an ω-standard model. Recall Definition 3 C • 2, that a model M is ω-standard iff $\mathbb{N}$ and $\mathbb{N}^{\mathsf{M}}$ are equal, although any model will contain a copy of $\mathbb{N}$ by Theorem 3 C • 3. To sum up, the W of Result 6 B • 1 has misinterpreted the coded notions of proof, ZFC, and so on through its misinterpretation of $\mathbb{N}$.

Because we've assumed $\mathrm{Con}(ZFC)$ of the real world natural numbers $\mathbb{N}$, for this model W, it must be that $\mathbb{N}^{\mathsf{W}} \neq \mathbb{N}$, that W has misinterpreted $\mathbb{N}$. In other words, W cannot be ω-standard. How does this misinterpretation help us in resolving the conflict? As noted above, there is a copy of $\mathbb{N}$ inside $\mathbb{N}^{\mathsf{W}}$. So if the coded proof of a contradiction were an actual natural number, we could decode the number to get an actual proof of a contradiction, and violate the assumption that ZFC is consistent. Hence this coded proof must correspond to a non-standard number, one which must come *after* the copy of the actual natural numbers. Hence the coded proof of inconsistency corresponds to a number which, from our perspective, is actually infinite. From **W**'s perspective, however, the number is finite. As such, we have no actual means of translating such a number into an actual proof.

This odd relation between the infinite and the finite is quite striking, and really shows the limits of first order logic to nail down these concepts properly. In a way, there's little difference between unboundedly large finite numbers and "infinite" numbers beyond them, as results like Compactness (2 D • 2) show. Even though we know such a code of inconsistency would correspond to an "infinite" number, such a number is finite from another perspective. Now to proceed to the next paradox in this line of thought, we need to introduce more results from set theory.

§ 6 D. Lévy–Montague Reflection

The next two paradoxes make use of the principle of reflection. Stated roughly, reflection is the idea that properties of the class of all sets can be witnessed by sets. This might sound trivial, since if the property holds in the entire universe, then it holds in any of the parts. But the statement is a little more subtle than that, since the property of being closed under the powerset operation is something true of the entire universe, but would require a very large set to satisfy this. The principle would then say that there is a set which is also closed under the powerset operation[iv]. But this is just a rough idea, and we need to develop these notions carefully.

The precise idea is given through the V_αs which compose the universe of sets under ZFC. Recall Theorem 3 A • 6 which says that $\mathsf{ZFC} \vdash \mathsf{V} = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$. The principle of reflection is the following theorem

6 D • 1. Theorem (Reflection)

Let φ be a formula. Therefore, $ZFC \vdash \exists\alpha\,(\varphi \leftrightarrow \langle V_\alpha, \acute{\in}\restriction V_\alpha\rangle \vDash \varphi)$.

Note that although $\acute{\in}$ is not a set in any model of ZFC, the restriction is:

$$ZFC \vdash \acute{\in}\restriction V_\alpha = \{\langle x, y\rangle : x \acute{\in} y \acute{\in} V_\alpha\} \subseteq V_\alpha \times V_\alpha.$$

So we can still talk about the structure $\langle V_\alpha, \acute{\in}\restriction V_\alpha\rangle^{\mathsf{M}}$ in any model $\mathsf{M} \vDash ZFC$. In particular, this shows that $ZFC \vDash \mathrm{Con}(\varphi)$ whenever $ZFC \vDash \varphi$. Note further that this is also a theorem of ZFC, when formulated in the proper way:

$$ZFC \vdash \forall\varphi\,(\mathrm{formula}(\varphi) \to ZFC \vdash \exists\alpha\,(\varphi \leftrightarrow \langle V_\alpha, \acute{\in}\restriction V_\alpha\rangle \vDash \varphi)).$$

§ 6 E. Reflection and formal compactness

The second paradox of this section highlights another disparity of the concepts we intend to encode in formulas, and the possible interpretations of these formulas. Stated roughly, it's not completely clear just how well we have coded the notions of formulas and consistency into ZFC. Although the concepts are equivalent for the real world, we unfortunately cannot conclude the equivalence everywhere else. In particular, there is a disparity between Compactness (2 D • 2) and Formal Compactness (2 E • 2) through a use of Gödel's Second Incompleteness Theorem (6 A • 3).

To begin, consider Reflection (6 D • 1). Although this is stated for a single formula, really it can be stated for any finite collection of formulas, because we can just take the conjunction and apply the result to that one formula: $\varphi_1 \wedge \cdots \wedge \varphi_n$. In essence, Reflection (6 D • 1) tells us that every finite subset of ZFC is known to be consistent by every model of ZFC. Naively appealing to compactness would yield that every model of ZFC knows ZFC is consistent in spite of Gödel's Second Incompleteness Theorem (6 A • 3).

[iv] or what it *thinks* is the powerset operation. The set may not contain all subsets.

6 E • 1. Result

The following paradoxical state of affairs holds:

1. for every finite subset $\Delta \subseteq ZFC$, $ZFC \vdash \mathrm{Con}(\Delta)$;
2. if $ZFC \vdash \forall \Delta \subseteq ZFC(|\Delta| < \aleph_0 \to \mathrm{Con}(D))$, then $ZFC \vdash \mathrm{Con}(ZFC)$; but
3. $ZFC \nvdash \mathrm{Con}(ZFC)$.

Proof .:.

1. By Reflection (6 D • 1), we can prove there is a model of Δ: $ZFC \vdash \exists M \exists \varepsilon(\langle M, \varepsilon\rangle \vDash \Delta)$. By Formal Completeness (2 E • 1) we may conclude $ZFC \vdash \mathrm{Con}(\Delta)$.
2. This holds by Formal Compactness (2 E • 2).
3. This holds by Second Incompleteness Theorem (6 A • 3). ⊣

The result calls into question just how well we've actually encoded statements like compactness as well as the notions of provability into ZFC, and further highlights the disparity between the meta-theory and the abilities of ZFC in proofs.

Just by looking at the statement of this result, we can conclude $ZFC \nvdash \forall \Delta \subseteq ZFC(|\Delta| < \aleph_0 \to \mathrm{Con}(\Delta))$. This is a similar issue that ω-standard models highlight: although I might show $\mathsf{PA} \vdash \varphi(n)$ for every $n \in \mathbb{N}$, it doesn't follow that $\mathsf{PA} \vdash \forall n \varphi(n)$. So the resolution to this idea is the same for Result 6 E • 1.

§ 6 F. Resolution to reflection and formal compactness

The resolution is that the codified ZFC isn't actually the same as ZFC. In fact, it's better to understand Reflection (6 D • 1) as a theorem scheme: for every real world formula φ, we get a theorem of ZFC. But it may be that a model $\mathsf{M} \vDash ZFC$ has more formulas in ZFC^{M} than in ZFC[v]. As a result, although it can confirm that every actual finite subset is consistent, it doesn't think that constitutes every finite subset, and so we haven't met the conditions of the formal compactness theorem. Part of the reason for this is due to ω-non-standard models. If M believes there are more natural numbers than there are, then it believes both that there are more formulas, and also that there are more finite collections of formulas, because its notion of "finiteness" has been twisted.

Of course if a model $\mathsf{M} \vDash \mathrm{Con}(ZFC)$, it will satisfy that all finite subsets of ZFC^{M} are consistent, regardless of whether it's ω-standard. So the issue is with models $\mathsf{W} \vDash \neg\mathrm{Con}(ZFC)$. As stated in Subsection 6 B, such models cannot be ω-standard. So while we can check the actual theorems of ZFC in the model, the model itself is so twisted that it doesn't even know what ZFC really is. That said, any misinterpretation of ZFC will contain the actual coded ZFC, $\{\varphi : \varphi \in ZFC\}$, just as any non-standard interpretation of $\mathbb{N}$ contains a copy of the actual $\mathbb{N}$. This leads into the next and final paradox due to Brice Halimi [2].

§ 6 G. Reflection and Gödel

One might interpret the first paradox of this section as saying that there are some models of ZFC which contain no models of ZFC. Indeed, it seems that Formal Completeness (2 E • 1) would imply this. But the issue brought up by this paradox is that every model of ZFC actually contains another model of ZFC. Now someone with some knowledge of set theory might not be too surprised at this, since there are classes like $\mathsf{L} \subseteq \mathsf{V}$ which will also model ZFC. The important thing to note is that the model guaranteed by the paradoxical result will be a set, meaning that the model recognizes that the set is a structure, and so properly understands when this set models a coded formula. Stated more basically, we have a conflict with the paradox of Subsection 6 B as well as Second Incompleteness Theorem (6 A • 3), since it would seem every model M thinks ZFC^{M} has a model.

Another way to interpret the paradox is that models of ZFC should be difficult to find if we consider Second Incompleteness Theorem (6 A • 3). But result tells us such models are in abundance: for every model, there is another model

[v] or rather in $\{\varphi^{\mathsf{M}} : \varphi \in ZFC\} \subseteq M$

inside it, and another model inside that. The models of set theory then act like Russian nesting dolls, except that there is no end to the chain of models[vi].

6 G • 1. Result

(ZFC) Every model $\mathsf{M} \vDash ZFC$ contains a set $\mathsf{W} \in M$ such that $\mathsf{W} \vDash ZFC$.

Proof .:.

This holds trivially if ZFC is inconsistent. So assume otherwise, and let $\mathsf{M} \vDash ZFC$.

If M is an ω-standard, then since $\mathrm{Con}(ZFC)$ holds of $\mathbb{N}$, it holds of $\mathbb{N}^{\mathsf{M}}$. Thus $\mathsf{M} \vDash \mathrm{Con}(ZFC)$ so by Formal Completeness (2 E • 1) there is a model $\mathsf{W} \in \mathsf{M}$ with $\mathsf{M} \vDash \mathsf{W} \vDash ZFC$. Since every $\varphi \in \mathsf{ZFC}$ has $\mathsf{M} \vDash \varphi \mathrel{\dot\in} ZFC$, $\mathsf{W} \vDash ZFC$.

So suppose M is ω-non-standard. By Theorem 3 C • 3, the real natural numbers $\mathbb{N}$ are an initial segment of $\mathbb{N}^{\mathsf{M}}$ and so the set $\{\varphi^{\mathsf{M}} : \varphi \in ZFC\} \subseteq \mathbb{N}^{\mathsf{M}}$ is an initial segment.

Claim 1

here is some non-standard $N \in \mathbb{N}^{\mathsf{M}}$ with $\mathsf{M} \vDash \mathrm{Con}(ZFC \cap N)$.

Proof .:.

Suppose not, and consider the set $\mathcal{N} = \{n \mathrel{\dot\in} \mathbb{N} : \mathrm{Con}(ZFC \cap n)\}^{\mathsf{M}} \in M$. By assumption, every non-standard natural number $N \in M$ has $\mathsf{M} \vDash \neg\mathrm{Con}(ZFC \cap N)$. So this collection $\mathcal{N} \subseteq \mathbb{N}$ is a subset of the standard natural numbers of M. But we know from Reflection (6 D • 1) that this set contains the standard natural numbers: $\mathbb{N} \subseteq \mathcal{N}$. Hence the standard natural numbers $\mathbb{N} = \mathcal{N} \in M$, contradicting that M is ω-non-standard. ⊣

Let $N \in M$ be as in the claim. Consider the collection $Z = (ZFC \cap N)^{\mathsf{M}} \in M$. Note that then $\mathsf{M} \vDash |Z| < \aleph_0$. Thus M believes there is a formula $F \in M$ which is the conjunction of all the formulas in Z. Keep in mind that $ZFC \subseteq Z$.

So by Reflection (6 D • 1), $\mathsf{M} \vDash \mathrm{Con}(F)$ and so $\mathsf{M} \vDash \mathrm{Con}(Z)$. By the same sort of argument as before, there is then a set $\mathsf{W} \in M$ where $\mathsf{M} \vDash \mathsf{W} \vDash Z$. So for each actual formula $\varphi \in ZFC$, $\mathsf{M} \vDash \mathsf{W} \vDash \varphi$. As a result, $\mathsf{W} \vDash \varphi$ for each $\varphi \in ZFC$ and so $\mathsf{W} \vDash ZFC$. ⊣

The issue here is even more confounding. In effect, if ZFC is consistent, it's *really* consistent. The issue—which is actually somewhat addressed in the proof—is that such an M as in this result doesn't necessarily think the corresponding $\mathsf{W} \vDash ZFC$ despite the fact that it does. What can account for the disparity, given that we've used that $\mathsf{M} \vDash \mathsf{W} \vDash ZFC$ implies $\mathsf{W} \vDash ZFC$ in the above proof?

§ 6 H. Resolving reflection and Gödel

In the case that M is ω-standard, M will realize $\mathsf{W} \vDash ZFC$, but otherwise M has again misinterpreted ZFC. That said, we can still find the actual coded version of ZFC from an outside perspective. But while M contains all these coded axioms, M can't isolate these axioms of ZFC; it needs to include non-standard formulas. And it is these non-standard formulas $\Phi \in M$ that prevent M from thinking W is a model of ZFC: $\mathsf{M} \vDash \mathsf{W} \nvDash \Phi$. In the case that M is ω-standard, M only contains codes of actual formulas, and so we have the result that $\mathsf{M} \vDash \mathsf{W} \vDash ZFC$ iff $\mathsf{W} \vDash ZFC$.

Note that even if M isn't ω-standard, it might still be the case that $\mathsf{M} \vDash \mathrm{Con}(ZFC)$. In fact, we can construct via Compactness (2 D • 2) a model M which is not ω-standard, and yet for any formula φ, φ is true in the real world

[vi] The use of 'chain' here is a bit tenuous, since the structure we're considering keeps changing: rather than $\mathsf{W} = \langle W, \varepsilon \rangle$ as it exists in M, we need to consider real-world version of W with universe $\hat{W} := \{w \in M : \mathsf{M} \vDash w \mathrel{\dot\in} W\}$ and relation $\hat{\varepsilon} := \{\langle u, w \rangle : \mathsf{M} \vDash \langle u, v \rangle \mathrel{\dot\in} \varepsilon\}$, forming $\hat{\mathsf{W}} = \langle \hat{W}, \hat{\varepsilon} \rangle$. This $\hat{\mathsf{W}}$—possibly $\notin M$—is an actual set and structure, isomorphic (loosely speaking) to $\mathsf{W} \in M$.

iff $\mathsf{M} \vDash \varphi$. Explicitly, rather than ZFC appended with the formulas $\{c > n : n \in \mathbb{N}\}$, we take the theory $\{\varphi : \varphi$ is true in the real world$\}$ appended with those formulas.

This is a result of the fact that ω-non-standard models do not contain $\mathbb{N}$ as a set despite the fact that the interpretation of $\mathbb{N}$ contains a copy of all the elements of $\mathbb{N}$. We've defined ω as the least limit ordinal, and clearly the real world $\mathbb{N}$ has this property. So if a model of ZFC contains $\mathbb{N}$ as a set, it is automatically an ω-standard model. As a result, ω-non-standard models can't determine where the real natural numbers end, and the non-standard numbers begin. So while such a model $\mathsf{M} \vDash \neg\,\mathrm{Con}(ZFC)$ places the real axioms of ZFC in a "finite" set, and finite collections of natural numbers have maximal elements, it doesn't believe the real ZFC has a maximal axiom or that $\mathbb{N}$ has a maximal number. It doesn't see the set of actual axioms ZFC, only subsets and extensions of this, that *will* have maximal elements according to M.

Appendix A. These Turn Out to be Paradoxes

As stated in Appendix A , this document is not a full list of the paradoxical results related to set theory. Instead this document is intended to present some such results whose resolution might be interesting to a more casual reader familiar with some logic. That said, there are well-known results called paradoxes which this document has not touched on, and are arguably the most appropriately called "paradoxes". In particular, this appendix covers three paradoxes which are intimately related: Russell's Paradox, the Burali–Forti paradox, and Cantor's Paradox. These three touch more on the foundations of set theory than the others, and don't really come from any particular axiomatization like ZFC. These paradoxes are more paradoxes for the meta-theory than any particular formulation or theory. All foundations will need some way of answering the conflict, and different axiomatizations take different positions. Because the resolutions are so connected, each paradox will be presented, and then all three will be discussed.

Section A1. *Three Paradoxes*

The first paradox to be discussed will be Russell's paradox, arguably the most famous of these paradoxes. The paradox itself was used to punch a hole into Frege's initial formulation of set theory, which might be considered "naive set theory" today. Russell's paradox was one of the first hurdles for set theory to overcome on its path to becoming a field of such importance in math and logic. Unlike the other paradoxes in Appendix A , the conclusion of Russell's paradox is a genuine contradiction. So any resolution will need to avoid the premises somehow. Note that we are no longer working in a mature, robust a system like ZFC, but instead relying more on our intuition and naive conceptions of sets.

A1 • 1. Result (Russell's Paradox)

It is contradictory to assume every property yields a set of members.

Proof .:.

Consider the property $\psi(x)$, defined to be the statement $x \notin x$. Suppose the set $A = \{x : \psi(x)\}$ exists. Now suppose $A \in A$. By definition of A, $\psi(A)$ and hence $A \notin A$.

Thus we must conclude $A \notin A$. Yet in this case, by definition, $\neg\psi(A)$ so that $A \in A$. Again, we have a contradiction. So either way, we have a contradiction, and hence the assumption is inconsistent. ⊣

A more intuitive way to put the theorem would be to say that not all collections are sets. Of course, the collection $\{x : x \notin x\}$ is somewhat ad hoc, but there are legitimately useful concepts that demonstrate the limits of set theory. To demonstrate this, the Burali–Forti paradox gives a particular example: the collection of all ordinals, Ord.

A1 • 2. Result (Burali–Forti)

(Naive set theory) The collection Ord of all ordinals is not a set.

Proof .:.

The definition of an ordinal is a transitive, $\in$-well ordered set, which means that on the set x, $\in$ is a well-founded, linear order, and x contains all its $\in$-predecessors. Now suppose Ord is a set. It's clear that $\in$ is a linear order, since for any distinct ordinals $\alpha, \beta \in \mathrm{Ord}$, $\alpha < \beta$ or $\beta < \alpha$, and the order $<$ on the ordinals is the same as $\in$. Ord can pretty easily be shown to be $\in$-well-founded, since all of its members are. Thus Ord is an ordinal, and so $\mathrm{Ord} \in \mathrm{Ord}$. But because the order of the ordinals $<$ is the same as $\in$, $\mathrm{Ord} < \mathrm{Ord}$, contradicting that $\mathrm{Ord} = \mathrm{Ord}$. ⊣

The proof of this theorem is a little more involved than is given here, because there are two notions of ordinal that one can use: one being an explicit construction like in ZFC, and the other being a notion of order-type. The same distinction can also be made for cardinals, though it matters less for our purposes. In fact, Cantor's paradox continues the line of thought from Burali–Forti (A1 • 2), but reduces the result to cardinals.

A1 • 3. Result (Cantor's Paradox)

(Naive set theory) There is no largest cardinal.

Proof .:.

Let κ be an arbitrary cardinal. Consider the powerset $\mathcal{P}(\kappa)$, and suppose $|\mathcal{P}(\kappa)| \leq \kappa$ so that there is a surjection $f : \kappa \to \mathcal{P}(\kappa)$. Consider the set $A = \{x \in \kappa : x \notin f(x)\}$. As a subset of κ, we must have $A \in \mathcal{P}(\kappa)$ and so $A \in \operatorname{im} f$. Thus there is some preimage $\lambda \in \kappa$ where $f(\lambda) = A$.

Suppose $\lambda \in A$. Thus $\lambda \in f(\lambda)$ by definition of λ. But by definition of A, $\lambda \notin f(\lambda)$, a contradiction. Thus $\lambda \notin A$, and hence $\lambda \in f(\lambda) = A$, another contradiction. Hence no such f can exist. And because there is an injection $g : \kappa \to \mathcal{P}(\kappa)$ by $\alpha \mapsto \{\alpha\}$, we have that $\kappa < |\mathcal{P}(\kappa)|$. Thus κ is not the largest cardinal. As κ was arbitrary, there can be no largest cardinal. $\dashv$

The idea that follows from this is that the collection of cardinals, $\{x : |x| = x\}$, cannot be a set, because otherwise the supremum of its elements—just the set theoretic union—would exist and be a cardinal, contradicting Cantor's Paradox (A1 • 3).

The proof of Cantor's Paradox (A1 • 3) is the same kind of proof as in Russell's Paradox (A1 • 1). One may take the issue to be that the power-set of a collection must be strictly bigger, and that this applies to the universe as well. So this makes two issues apparent: one about how large we can make sets, and one about how many smaller sets we can make.

Section A2. *Unsatisfying Resolutions*

All of these three paradoxes revolve around two issues to take stances on, and demonstrate a tension between the ability to build bigger sets, and to restrict to smaller subsets. Firstly, we will look at how a more traditional set theory, ZFC, addresses these issues, and then how a more eccentric set theory looks at them.

§ A2 a. Traditional set theory

Russell's paradox works off of two assumptions: that you can build sets, and that you can consider (definable) subsets. ZFC allows via the axiom of comprehension to consider (definable) subsets, and so avoids Russell's Paradox (A1 • 1) by restricting the the size of sets we can build. In particular, the axiom of foundation rules out (finite) $\in$-loops like $x \in x$. As a result, in ZFC, the collection $\{x : x \notin x\}$ is just the entire universe. And the universe V is not a set for ZFC again because of the axiom of foundation: we can't have the finite $\in$-loop $V \in V$. So although all the elements exist, ZFC does not allow you to collect these elements together. Burali–Forti (A1 • 2) and Cantor's Paradox (A1 • 3) also get similar treatments as Russell's Paradox (A1 • 1) from ZFC: because we can consider these problem subsets, we can't build too high for fear of the set theoretic universe collapsing under a contradiction.

But this hasn't been too much of a problem for the rest of this document, since we can call such collections '*classes*' rather than 'sets' as in von Neumann–Bernays–Gödel, Morse–Kelley, and other set theories. In this way, we can still consider such collections, and reason about them, but must recognize their difference from sets.

Unfortunately, this line of reasoning is somewhat unsatisfactory, since it effectively dodges Russell's Paradox (A1 • 1) by saying "oh, that collection is not a *set*, it's uh... a *class*" without meaningfully distinguishing what makes the two notions distinct[i]: both are just collecting individual things together. Worse yet, we can again apply the reasoning of Russell's Paradox (A1 • 1) to classes instead of sets and again get the same result. The response is then to move to 2-classes; a level above classes, and two levels above sets. Again we run into the same paradox, and so introduce 3-classes, then 4-classes, and so on. The result is an iterated hierarchy of collections. The never-ending process of creating collections, the open top on Figure 3 A • 8, forces us to adopt this kind of view, but doesn't satisfactorily say how the levels really differ. And whatever general notion of collection we come up with to unite the entire iterated hierarchy, we can again apply Russell's Paradox (A1 • 1) to that notion to end up with a contradiction. In short, you can't put everything in a box with this framework.

But this set theoretic ideology isn't the only one around; it's just the most widely accepted and usable framework for mathematics. There are other systems which can give different responses to these paradoxes.

§ A2 b. New Foundations

Quine's New Foundations (NF) is strange for a number of reasons. For example, rather than having a well-foundedness of membership, as in ZFC, there is a kind of well-foundedness of the defining formulas. Moreover, it's finitely axiomatizable; the axiom of choice is false in it; and any model contains itself: $V \in V$. Note that although a modified, fairly different system NFU has been shown to be consistent relative to ZFC [3], NF itself is not known to be consistent relative to any normal mathematical system. And as a system less studied, it's not entirely clear just how well it has avoided the contradictions of these paradoxes, although it has blocked the usual approaches.

Regardless, the usual approach to Russell's Paradox (A1 • 1) does not apply to NF. Rather than saying that the collection is everything and not a set, NF says that we cannot collect elements together if we don't do it properly. In particular, the defining formula must be *stratified*.

A2 b • 1. Definition

Let φ be a formula in the signature $\{\acute{\in}\}$ with variables among $v_0, ..., v_n$. φ is stratified iff there is a way to arrange $v_0, ..., v_n$ such that
(1) if $v_i \acute{\in} v_j$ occurs in φ, then v_i *directly* precedes v_j in the arrangement; and
(2) if $v_i = v_j$ occurs in φ, then v_i shares the same place as v_j in the arrangement.

"Sharing the same place" will be denoted by being within parantheses. For example, $x \acute{\in} y \wedge x \acute{\in} z$ is stratified as witnessed by the arrangement $\langle x, (y, z)\rangle$, where y and z share the same place. However, $\neg x \acute{\in} x$ is not stratified, since this would require x to precede itself. So while we can form the set $V = \{x : x = x\}$[ii], the subset $\{x \in V : x \notin x\} \subseteq V$ isn't guaranteed to exist by the axioms of NF. Hence NF gets around Russell's Paradox (A1 • 1) by saying "you can't define a set in that way" instead of "such a set doesn't exist".

As noted in this resolution, unlike with ZFC, NF allows us to put everything—or everything it can see—into a box: $V \in V$. In this way, there would be a closed top on Figure 3 A • 8. But then how do we avoid the paradoxes of Burali–Forti (A1 • 2) and Cantor's Paradox (A1 • 3), which deal with reaching the top of this diagram?

As noted before, the proof of Burali–Forti (A1 • 2) relies on a certain characterization of "ordinal" that doesn't hold in NF, namely that the order type of an ordinal can be identified with the membership relation. Trying to define a function mapping an ordinal α to its order-type $T(\alpha)$[iii] results in a function T that NF doesn't think is a set because of the paradox.

The proof of Cantor's Paradox (A1 • 3) is similar to Russell's Paradox (A1 • 1)[iv]. The way NF gets around Cantor's Paradox, however, is slightly different. Cardinality and ordinals in NF don't look the same as in ZFC. In fact, because

[i] At least in a way that is intuitively obvious or clear of the real world.
[ii] $x = x$ is stratified as witnessed trivially by $\langle x\rangle$
[iii] I'll note I haven't been precise on just what the ordinal or order-types are in NF, because they don't have the same construction as in ZFC. But this document isn't about NF, and too much detail introducing an entirely different theory would be uncalled for.
[iv] although really it's Russell who copied Cantor [1]

of how strict NF is on how sets are defined—needing to be stratified—depending on how you define what an ordered pair is, you get different notions of cardinality. And these different notions can differ on whether two sets have the same cardinality or not, regardless of whether the defining formulas are stratified [4]. Even if we focus on a particular notion of ordered pair, the existence of functions and sets like $\{x : x \notin f(x)\}$ is called into question by their defining formula again. Even the supposedly trivial injection $x \mapsto \{x\}$ from V to $\wp(V)$ isn't guaranteed to exist because the defining formula isn't stratified. Worse yet, the requirement of stratification means we can't compare the cardinality of a set X and its powerset $\wp(X)$ directly. Instead we need to consider a question—which would seem to be equivalent in the meta-theory—of how $\{\{x\} : x \in X\}$ and $\wp(X)$ compare. But again, it's not clear from NF's perspective that X has the same cardinality as $\{\{x\} : x \in X\}$.

The result is that NF, while an interesting system, is very unintuitive and unconducive to doing math compared to a system like ZFC. And ZFC, while giving somewhat concrete answers to these paradoxes, doesn't substantively address some of the concerns around collections in general, resulting in a hierarchy of kinds of collections. At best the collection of *everything* would have to be viewed from a ZFC ideology only as a kind of "potential" collection that can't actually exist despite the fact that all its members do. NF, on the other hand, allows us to consider such collections, but fundamentally fails to grasp the meaning of well formed ideas like that of cardinality.

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